6. Thus, in the present article methods of solution have been given for the system (6)-(7) which enable one to find temperature fields and fluxes.

With the aid of these methods, approximation formulas have been found for the main characteristics of a thermoelement: the heat flux and the internal resistance.

It has been shown that in the previously proposed formulas for the heat flux a quadratic (in the Thomson effect) term is missing, and in the resistance a linear (in the current term) is also missing. The estimates show that these effects may result in the deviation of the volt—ampere characteristics from linear of several percent.

NOTATION

T, y, u, temperatures; q, Q, Θ, heat fluxes; x, ξ, η, coordinates; ϰ, Λ, thermal conductivities; ρ, γ, resistivities; α, coefficient of thermo-emf; τ, β, Thomson coefficients; L, length; S, cross section; J, current density; I, total current; E, electric field intensity; W, power; V, voltage.

LITERATURE CITED

1. L. D. Landau and E. M. Lifshits, Electrodynamics of Continuous Media, Addison-Wesley (1960).
2. A. G. Samoilovich and L. L. Korenblit, UFI, 10, 9, No. 2, 243 (1953).†
3. A. I. Burshtein, Physical Basis for Evaluation of Semiconductor Thermoelectric Devices [in Russian], Fizmatgiz, Moscow (1962).
4. B. Ya. Moizhes, Fiz. Tverd. Tela, 2, No. 4 (1960).
5. L. S. Okhotin, A. A. Efremov, V. S. Okhotin, and A. S. Pushkarskii, Thermoelectrical Generators [in Russian], Atomizdat, Moscow (1971).
6. S. Friedman, in: Direct Energy Transformation [Russian translation], Mir, Moscow (1969).
7. A. F. Ioffe, L. S. Stil'bans, E. K. Iordanishvili, and T. S. Stavitskaya, Thermoelectrical Cooling [in Russian], Izd. Akad. Nauk SSSR, Moscow (1956).
8. Yu. I. Tsvetkov, Izv. Vuzov, Énergetika, No. 4 (1974).

†As in Russian original.

APPLICATION OF INFINITE SYSTEMS TO THE SOLUTION OF BOUNDARY-VALUE PROBLEMS OF STEADY THERMAL CONDUCTION IN NONUNIFORM MEDIA

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A method of calculating the temperature field in nonuniform media is described. Examples of the calculation of the temperature distribution for an exponential variation of the thermal conductivity of the medium and also in a multilayer structure are presented.

In the rectangular region $\Omega\{0 \leqslant x \leqslant l,\ 0 \leqslant y \leqslant 1\}$ we will consider the boundary-value problem of steady thermal conduction [1]

$$\frac{\partial}{\partial x}\left[h(x)\ \frac{\partial u}{\partial x}\right] + h(x)\frac{\partial^2 u}{\partial y^2} - q^2 u = -f(x,\ y), \qquad (1)$$

$$u = 0 \text{ on } \partial\Omega, \qquad (2)$$

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where $h(x) > 0$ is the thermal conductivity, $f(x, y) > 0$ is the density of sources at the point $M(x, y)$, and q^2 = const is the coefficient of volume absorption of heat.

We will represent the solution $u(x, y)$ of Eq. (1) which satisfies the boundary condition (2) for $y = 0$ and $y = 1$ in the form

$$u(x, y) = \sum_{s=1}^{\infty} v_s(x) \sin s\pi y, \tag{3}$$

in which the function $v_s(x)$ is found from the self-conjugate second-order differential equation

$$\frac{d}{dx}\left[h(x)\frac{dv_s}{dx}\right] - (s\pi)^2 h(x) v_s(x) - q^2 v_s(x) = -f_s(x), \tag{4}$$

$$f_s(x) = 2\int_0^1 f(x, \xi) \sin s\pi\xi d\xi.$$

According to Eq. (2), the solution $v_s(x)$ of Eq. (4) must satisfy the conditions

$$v_s(0) = v_s(l) = 0. \tag{5}$$

We will seek the solution $v_s(x)$ of the boundary-value problem (4) and (5) in the class of continuous functions $C(0, l)$ in the following form:

$$v_s(x) = \sum_{k=1}^{\infty} a_k^s \sin\frac{k\pi x}{l}. \tag{6}$$

Expanding the functions $h(x)$ and $f_s(x)$ in the range $0 < x < l$ in Fourier series, we obtain

$$h(x) = \frac{\vartheta_0}{2} \sum_{k=1}^{\infty} \vartheta_k \cos\frac{k\pi x}{l}, \tag{7}$$

$$f_s(x) = \sum_{k=1}^{\infty} v_k^s \sin\frac{k\pi x}{l}, \tag{8}$$

where

$$\vartheta_k = \frac{2}{l}\int_0^l h(\xi)\cos\frac{k\pi\xi}{l}d\xi,$$

$$v_k^s = \frac{2}{l}\int_0^l f_s(\xi)\sin\frac{k\pi\xi}{l}d\xi.$$

Since the function $v_s(x)$ $C(0,\epsilon l)$ and satisfies condition (5), it follows from expansion (6) that the series

$$\sum_{k=1}^{\infty} a_k^s \frac{k\pi}{l}\cos\frac{k\pi x}{l}$$

converges and

$$\frac{dv_s}{dx} = \sum_{k=1}^{\infty} a_k^s \frac{k\pi}{l}\cos\frac{k\pi x}{l}.$$

We will introduce the auxiliary functions

$$\Phi(x) = h(x)\frac{dv_s}{dx}, \quad \Psi(x) = h(x)\, v_s(x). \tag{9}$$

Assuming that for $h(x)$, $v_s(x)$, and dv_s/dx, belonging to the space $L_2(0, l)$ of functions integrable with a square, there is a general equation of closure [2], we will write the expansions $\Phi(x)$ and $\Psi(x)$ in Fourier series:

$$\Phi(x) = \sum_{k=0}^{\infty} \Phi_k \cos\frac{k\pi x}{l},$$

$$\Psi(x) = \sum_{k=1}^{\infty} \Psi_k \sin \frac{k\pi x}{l}. \tag{10}$$

Then for the coefficients Φ_k and Ψ_k using the rule of multiplication of Fourier series we can write the equations [2]

$$\begin{aligned} \Phi_k &= \frac{1}{2} \sum_{j=1}^{\infty} a_j^s \frac{j\pi}{l} (\vartheta_{k-j} + \vartheta_{k+j}), \quad k = 0, 1, 2, \ldots, \\ \Psi_k &= \frac{1}{2} \sum_{j=1}^{\infty} a_j^s (\vartheta_{k-j} - \vartheta_{k+j}), \quad k = 1, 2, \ldots \end{aligned} \tag{11}$$

Using Eq. (9) and substituting the expansions (6), (7), and (10) into Eq. (4), we obtain

$$\frac{k\pi}{l} \Phi_k + (s\pi)^2 \Psi_k + q^2 a_k^s = \nu_k^s, \quad k = 1, 2, \ldots . \tag{12}$$

Finally, in view of Eq. (11), relation (12) is transformed to the following infinite system of linear algebraic equations of the second kind with respect to the required coefficients:

$$q^2 a_k^s + \sum_{j=1}^{\infty} a_j^s \frac{\pi^2 kj}{2l^2} (\vartheta_{k-j} + \vartheta_{k+j}) + \sum_{j=1}^{\infty} a_j^s \frac{\pi^2 s^2}{2} (\vartheta_{k-j} - \vartheta_{k+j}) = \nu_k^s, \tag{13}$$

where s occurs in the equations of this system as a parameter. In addition, for the Fourier coefficients ϑ_n of the function h(x) in (13) we must assume that $\vartheta_n = \vartheta_{-n}$.

It will be shown below that the infinite system (13) can be transformed in such a way that its matrix operator in the class of functions h(x) considered is Fredholm. In this way we will establish that the solvability of the infinite system, and, consequently, the solution of the initial boundary-value problem (1) and (2), in accordance with expansions (3) and (6), can be represented in the form

$$u(x, y) = \sum_{s=1}^{\infty} \sum_{k=1}^{\infty} a_k^s \sin \frac{k\pi x}{l} \sin s\pi y.$$

Denoting the matrix elements of the infinite system (13) by A_{kj},

$$A_{kj} = \frac{\pi^2}{2} \left\{ \frac{kj}{l^2} (\vartheta_{k-j} + \vartheta_{k+j}) + s^2 (\vartheta_{k-j} - \vartheta_{k+j}) \right\}, \tag{14}$$

we transform this system to the form

$$a_k^s + \sum_{\substack{j=1 \\ j \neq k}}^{\infty} \frac{\pi^2 kj}{2l^2 \omega_k} (\vartheta_{k-j} + \vartheta_{k+j}) a_j^s - \sum_{\substack{j=1 \\ j \neq k}}^{\infty} \frac{\pi^2 s^2}{2\omega_k} (\vartheta_{k-j} - \vartheta_{k+j}) a_j^s = \frac{\nu_k^s}{\omega_k}, \tag{15}$$

where $\omega_q = q^2 + A_{kk}$ and $\omega_k = O(k^2)$ as $k \to \infty$.

We will introduce the notation

$$B_k = \frac{\nu_k^s}{\omega_k}, \quad P_{kj} = \frac{\pi^2 kj}{2l^2 \omega_k} (\vartheta_{k-j} + \vartheta_{k+j}),$$

$$Q_{kj} = \frac{\pi^2 s^2}{2\omega_k} (\vartheta_{k-j} - \vartheta_{k+j}),$$

after which system (15) and, of course, the initial infinite system (13) take the form

$$\begin{aligned} & a_k^s + \sum_{j=1}^{\infty} R_{kj} a_j^s = B_k, \quad k = 1, 2, \ldots, \\ & R_{kj} = \begin{cases} P_{kj} + Q_{kj} & \text{for } k \neq j \\ 0 & \text{for } k = j. \end{cases} \end{aligned} \tag{16}$$

We will investigate the possibility that a solution of the infinite system (16) exists and we will obtain it by the reduction method.

We will first show that for the matrix elements P_{kj} the following estimate holds:

$$\sum_{k,j=1}^{\infty} |P_{kj}|^2 < \infty. \tag{17}$$

In fact

$$\sum_{k,j=1}^{\infty} |P_{kj}|^2 \leqslant \frac{\pi^2}{l^2} \sum_{k=1}^{\infty} \left(\frac{k}{\omega_k}\right)^2 \sum_{j=1}^{\infty} j^2 (|\vartheta_{k-j}|^2 + |\vartheta_{k+j}|^2).$$

Since

$$\sum_{j=1}^{\infty} j^2 |\vartheta_{k-j}|^2 \leqslant \text{const} \sum_{j=1}^{\infty} |(k-j)\,\vartheta_{k-j}|^2 \leqslant \text{const} \cdot \sum_{m=-\infty}^{k-1} |m\vartheta_m|$$

$$\sum_{j=1}^{\infty} j^2 |\vartheta_{k+j}|^2 \leqslant \sum_{j=1}^{\infty} |(k+j)\,\vartheta_{k+j}|^2 = \sum_{m=k+1}^{\infty} |m\vartheta_m|^2,$$

we have

$$\sum_{k,j=1}^{\infty} |P_{kj}|^2 \leqslant \text{const} \sum_{k=1}^{\infty} \frac{1}{k^2} \sum_{m=1}^{\infty} |m\vartheta_m|^2. \tag{18}$$

If the function h(x) is continuous in the interval $0 < x < l$, then at least

$$\vartheta_m = O(m^{-2}) \quad \text{as} \quad m \to \infty.$$

In this case the series $\sum_{m=1}^{\infty} |m\vartheta_m|^2$ converges and, consequently, the binary series (18) also converges, i.e., estimate (17) holds.

Similarly for Q_{kj} the following inequality holds:

$$\sum_{k,j=1}^{\infty} |Q_{kj}|^2 < \infty. \tag{19}$$

But then using (17) and (19) for the matrix elements of the infinite system (16) we have the estimate

$$\sum_{k,j=1}^{\infty} |R_{kj}|^2 \leqslant \sum_{k,j=1}^{\infty} (|P_{kj}| + |Q_{kj}|)^2 \leqslant \text{const} \left(\sum_{k,j=1}^{\infty} |P_{kj}|^2 + \sum_{k,j=1}^{\infty} |Q_{kj}|^2 \right) < \infty. \tag{20}$$

In addition, it is obvious that the sequence of free terms of system (16) satisfies the condition

$$\sum_{k=1}^{\infty} |B_k|^2 < \infty. \tag{21}$$

The estimate (20) enables us to establish [3] that the matrix of system (16) generates in Hilbert space l^2 of the sequences $\{a_k^s\}_{k=1}$ a completely continuous operator. Since, according to (21), the columns of free terms also belong to l^2, in view of the Hilbert theorem for an infinite system with completely continuous form we have the following alternative: either this system has a unique solution which satisfies the condition

$$\sum_{k=1}^{\infty} |a_k^s|^2 < \infty,$$

i.e., belonging to l^2, or in the uniform system corresponding to it there is a solution which differs from zero in the same space.

The absence of a nontrivial solution of the homogeneous system is easily established as a consequence of the linear independence of the rows of the system matrix, which proves the existence and uniqueness of the solution of the homogeneous system (16) for the case when $h(x) \in C\ (0, l)$.

In this case, to solve the infinite system (16) we will use the method of reduction, which means that approximate values of a_k^s can be found from the truncated system

$$\bar{a}_k^s + \sum_{j=1}^{N} R_{kj} \bar{a}_j^s = B_k, \quad k = 1, 2, \ldots . \tag{22}$$

Note that when the function h(x) is piecewise smooth, having discontinuities of the first kind at certain points $x_i \in (0, l)$, i.e., when for the Fourier coefficients only the relation

$$\vartheta_m = O(m^{-1}) \quad \text{as} \quad m \to \infty \tag{23}$$

is satisfied, the series $\sum_{m=1}^{\infty} |m\vartheta_m|$ diverges and there is no upper estimate (18) for the matrix element P_{kj}. In this case the basis of the reduction of the infinite system (16) requires a separate investigation.

We write the system (16) in the form

$$a_k^s + \sum_{\substack{j=1 \\ j\neq k}}^{\infty} a_j^s \frac{\pi^2 j}{2l^2} \cdot \frac{k}{\omega_k} (\vartheta_{k-j} + \vartheta_{k+j}) + \sum_{\substack{j=1 \\ j\neq k}}^{\infty} a_j^s \left(1 - \frac{\pi^2 j}{2l^2}\right) \frac{\pi^2 s^2}{2\omega_k} \frac{\vartheta_{k-j} - \vartheta_{k+j}}{1 - \dfrac{\pi^2 j}{2l^2}} = B_k. \tag{24}$$

We will introduce the following notation:

$$X_k = \frac{\pi^2 k}{2l^2} a_k^s, \quad Y_k = \left(1 - \frac{\pi^2 k}{2l^2}\right) a_k^s,$$

$$C_{kj} = \frac{k}{\omega_k} (\vartheta_{k-j} + \vartheta_{k+j}),$$

$$D_{kj} = \frac{\pi^2 s^2}{2\omega_k} \frac{\vartheta_{k-j} - \vartheta_{k+j}}{1 - \dfrac{\pi^2 j}{2l^2}} .$$

Taking into account the fact that $a_k^s = X_k + Y_k$, system (24) takes the form

$$X_k + Y_k + \sum_{\substack{j=1 \\ j\neq k}}^{\infty} C_{kj} X_j + \sum_{\substack{j=1 \\ j\neq k}}^{\infty} D_{kj} Y_j = B_k. \tag{25}$$

We will consider two auxiliary infinite systems in terms of the unknown quantities α_k and β_k:

$$\begin{aligned} &\alpha_k + \sum_{j=1}^{\infty}{}' C_{kj} \alpha_j = \varepsilon_k B_k, \quad k = 1, 2, \ldots, \\ &\beta_k + \sum_{j=1}^{\infty}{}' D_{kj} \beta_j = (1 - \varepsilon_k) B_k, \quad k = 1, 2, \ldots . \end{aligned} \tag{26}$$

Here in the sum Σ' we have omitted the term with j = k, and ε_k have been chosen so that the systems (26) are satisfied identically when $\alpha_k = (\pi^2 k/2l^2) a_k^s$ and $\beta_k = [1 - (\pi^2 k/2l^2)] a_k^s$, where a_k^s is the solution of the infinite system (24).

It is easy to show that when conditions (23) are satisfied the following estimates hold for C_{kj} and D_{kj}:

$$\sum_{k,j=1}^{\infty}{}' |C_{kj}|^2 < \infty, \quad \sum_{k,j=1}^{\infty}{}' |D_{kj}|^2 < \infty.$$

Whence, when there are no nontrivial solutions of the homogeneous system corresponding to (26), there follows the existence and uniqueness of solutions of the inhomogeneous infinite system (26) which satisfy the conditions

$$\sum_{k=1}^{\infty} |\alpha_k|^2 < \infty, \quad \sum_{k=1}^{\infty} |\beta_k|^2 < \infty. \tag{27}$$

In addition, to solve infinite system (26) we will use the method of reduction.

Suppose α_k and β_k are, respectively, solutions of the infinite systems of (26). Then system (25) can be satisfied by assuming

$$X_k = \alpha_k, \quad Y_k = \beta_k.$$

Therefore,

$$a_k^s = \alpha_k + \beta_k \tag{28}$$

is a solution of the infinite system (25), and, consequently, its equivalent (16) also, and in view of (27) this solution belongs to the space l^2. It should be noted that the solution $\bar{a}_s^k$ of the truncated system (22), corresponding to the infinite system (16), can be represented as

$$\bar{a}_k^s = \bar{\alpha}_k + \bar{\beta}_k,$$

where $\bar{\alpha}_k$ and $\bar{\beta}_k$ are, respectively, the solutions of the truncated systems

$$\bar{\alpha}_k + \sum_{j=1}^{N}{}' C_{kj}\bar{\alpha}_j = \varepsilon_k B_k, \quad k = 1, 2, \ldots, N,$$

$$\bar{\beta}_k + \sum_{j=1}^{N}{}' D_{kj}\bar{\beta}_j = (1-\varepsilon_k) B_k, \quad k = 1, 2, \ldots, N.$$

It follows from the applicability of reduction for the infinite systems (26) that

$$\bar{\alpha}_k \to \alpha_k, \quad \bar{\beta}_k \to \beta_k \quad \text{as} \quad N \to \infty.$$

Then, taking (28) into account, we obtain as $N \to \infty$

$$\bar{a}_k^s = \bar{\alpha}_k + \bar{\beta}_k \to \alpha_k + \beta_k = a_k^s.$$

This indicates the possibility of reducing the infinite system (16) for the case of a discontinuous piecewise-smooth function h(x).

Hence, the rigorous solution of the boundary-value problem (1) and (2) can be represented in the form of the double trigonometric series (14) with coefficients a_k^s, which satisfies the infinite system (16), where we use the method of reduction to solve this system.

The form of the solution obtained enables one to calculate the steady temperature field in important practical problems with varying thermal conductivity. To carry out the calculation using the above scheme it is only necessary to assign the Fourier coefficients of the functions h(x) and f(x, y). We will consider examples of these calculations.

1. Suppose the thermal conductivity varies as follows:

$$h(x) = A\exp\left(-\frac{r}{l}x\right), \quad A, r - \text{const.} \tag{29}$$

The problem will be solved for the function f(x, y) corresponding to a heat source concentrated at a certain point $M_0(x_0, y_0) \in \Omega$

$$f(x, y) \equiv f_* = Q\delta(M - M_0), \quad Q = \text{const} > 0.$$

In this case

$$\vartheta_k = 2Arl\frac{1-(-1)^k \exp(-r)}{r^2 + k^2\pi^2 l^2}, \quad k = 0, 1, 2\ldots,$$

$$v_k^s = \frac{4Q}{l}\sin\frac{k\pi x_0}{l}\sin s\pi y_0, \quad k = 1, 2, \ldots; \ s = 1, 2, \ldots,$$

and the solution (14) of the initial boundary-value problem, having the form

$$u = G(x, y, x_0, y_0),$$

is fundamental for an exponential variation of the thermal conductivity of the medium (29). This fundamental solution in view of the linearity of Eq. (1) enables one to write the solution with an arbitrary right side:

$$u(x, y) = \frac{1}{Q}\iint_{\Omega} G(x, y, x_0, y_0) f(x_0, y_0)\, dx_0 dy_0.$$

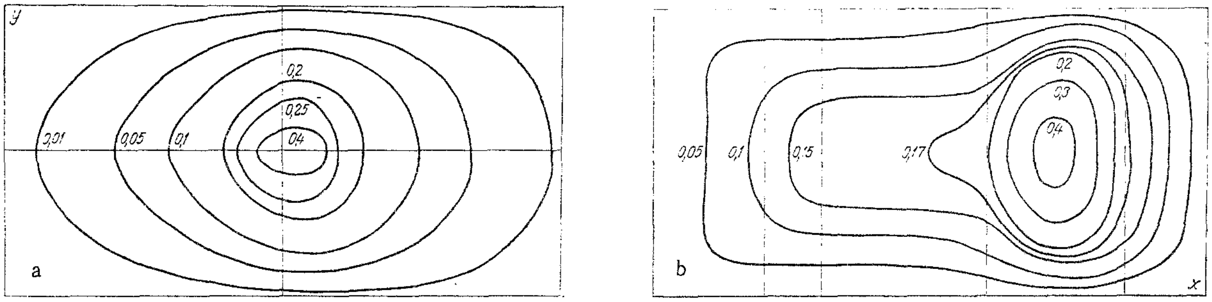

Fig. 1. Temperature field: a) in a nonuniform medium; b) in a multilayer structure.

Figure 1a shows isotherms of the temperature field of the fundamental solution for the following values of the parameters: A = 1, Q = 1, r = ln 10; $l = 2$, $x_0 = {}^1/_2$. and $y_0 = {}^1/_2$.

2. The possibility of obtaining a solution of the boundary-value problem (1) and (2) using this method in the case of a piecewise-smooth function h(x) enables us to calculate the temperature distribution in a multilayer structure when there is no contact resistance at the surfaces of the touching layers.

This formulation of the problem corresponds to assigning h(x) in the form of a piecewise-constant function

$$h(x) = \begin{cases} H_1, & 0 = x_0 < x < x_1, \\ H_2, & x_1 < x < x_2, \\ \cdot \; \cdot \; \cdot \; \cdot \; \cdot \; \cdot \; \cdot \; \cdot \; \cdot \\ H_n, & x_{n-1} < x < x_n = l, \end{cases} \tag{30}$$

where H_i = const > 0, i = 1, 2, ..., n. In this case the Fourier coefficients of this function can be calculated from the equation

$$\vartheta_k = \frac{2}{\pi k} \sum_{i=1}^{n} H_i \left(\sin \frac{k\pi x_i}{l} - \sin \frac{k\pi x_{i-1}}{l} \right). \tag{31}$$

Figure 1b represents isotherms of the temperature field in a layered structure in which the thermal conductivity varies as given by Eq. (30) for n = 5, $l = 2$, $x_1 = 0.1l$, $x_2 = 0.3l$, $x_3 = 0.6l$, and $x_4 = 0.85l$ for $H_1 = 0.8$, $H_2 = 0.2$, $H_3 = 1$, $H_4 = 0.1$, $H_5 = 0.5$, and q = 0. The calculation was carried out in this case assuming uniformly distributed thermal sources with a constant density f(x, y) ≡ 1.

Note that a feature of this method of solution is the possibility of carrying out calculations in a multilayer structure for any number of layers without complicating the amount of computational work, since the number of layers n is only taken into account in Eq. (31) for the Fourier coefficients of the piecewise-constant function h(x).

In conclusion, we note that using this method one can obtain a solution of Eq. (1) with mixed boundary conditions on ∂Ω.

In particular, if

$$u = 0 \text{ on } x = 0 \text{ and } x = l, \quad \frac{\partial u}{\partial y} = 0 \text{ on } y = 0 \text{ and } y = 1, \tag{32}$$

the solution of the boundary-value problem (1), (32) can be represented in the form

$$u(x, y) = \sum_{s=0}^{\infty} \sum_{k=1}^{\infty} a_k^s \sin \frac{k\pi x}{l} \cos s\pi y$$

with expansion coefficients a_k^s which satisfy an infinite system of the form (13) in which the free terms ν_k^s are calculated from the equation

$$\nu_k^s = \frac{4}{l} \int_0^l \int_0^1 f(\xi, \eta) \sin \frac{k\pi\xi}{l} \cos s\pi\eta \, d\xi d\eta.$$

Similarly, one can solve the mixed boundary-value problem with conditions of the first kind on y = 0 and y = 1 and of the second kind on the other sections x = 0 and x = l of the boundary of the region.

LITERATURE CITED

1. A. N. Tikhonov and A. A. Samarskii, Equations of Mathematical Physics [in Russian], Nauka, Moscow (1966).
2. G. M. Fikhtengol'ts, Course on Differential and Integral Calculus [in Russian], No. 3, Fizmatgiz, Moscow (1963).
3. L. V. Kantorovich and V. I. Krylov, Approximate Methods of Higher Analysis [in Russian], Fizmatgiz, Moscow (1962).